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The Interrupted Poisson Process As An Overflow Process

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Traffic overflowing a first-choice trunk group can be approximated accurately by a simple renewal process called an interrupted Poisson process–a Poisson process which is alternately turned on for an exponentially distributed time and then turned off for another (independent) exponentially distributed time. The approximation is obtained by matching either the first two or three moments of an interrupted Poisson process to those of an overflow process. Numerical investigation of errors in the approximation and subsequent experience has shown that this method of generating overflow traffic is accurate and very useful in both simulations and analyses of traffic systems.

I. INTRODUCTION

A Poisson process which is alternately turned on for an exponentially distributed time and then turned off for anotber (independent) exponentially distributed time will be called an *interrupted* Poisson process–it can be viewed as a Poisson process modulated by a random switch. It was suggested by W. S. Hayward of Bell Laboratories tbat such a process be used to simulate overflow traffic. We will sbow tbat tbe interrupted Poisson process provides a simple and accurate method of simulating overflow traffic.

The objective is to reduce the cost of computer simulations of traffic systems by using tbe interrupted Poisson process to model tbe overflow traffic. Generation of actual overflow traffic by simulating tbe behavior of the trunk group from which it overflows is time consuming since a record must be kept of *all* calls wbicb are offered to the subtending trunk group, whether they contribute to the overflow traffic or not. This is especially true when the traffic is overflowing a large trunk group. Moreover, the interrupted Poisson process provides a simple, approximate description of the overflow traffic and consequently facilitates analytical studies.

This method of generating overflow traffic has been used successfully in many studies of traffic systems. Examples of its application can be found in Refs. 1, 2, and 3 and also in numerous unpublished works. Since the method has been found to be very useful and requests for wider dissemination have been received by the author, this paper has been prepared.

In Section II we derive the distribution of the number of busy trunks in an infinite trunk group when the offered traffic is generated by an interrupted Poisson process. The corresponding distribution for an overflow input has been computed by Kosten.[4] Now, matching the first three moments of the two distributions, we obtain equations for the parameters of the interrupted Poisson process. We also give these equations for a two-moment match. The errors committed in approximating the distribution given by Kosten are also examined. In Section III we derive the interarrival time distribution for a traffic stream generated by an interrupted Poisson process.

II. MOMENT-MATCH EQUATIONS

Let the interrupted Poisson traffic be offered to an infinite trunk group. Let λ be the intensity of the Poisson process, $1/\gamma$ be the mean on-time of the random switch, $1/\omega$ be the mean off-time, and $1/\mu$ be the mean service time. Let the state of the system be described by (m, n) with state probabilities $p(m, n)$ where m is the number of servers busy, and n is the state of the switch taking on the value of 1 or 0 according to whether the process is on or off.

The equilibrium equations for the stationary state probabilities are

$$
\begin{aligned}
(m\mu + \omega)p(m, 0) &= \gamma p(m, 1) + (m + 1)\mu p(m + 1, 0), \qquad m \geqq 0,\\
(m\mu + \gamma + \lambda)p(m, 1) &= \omega p(m, 0) + (m + 1)\mu p(m + 1, 1)\\
&\qquad + \lambda p(m - 1, 1), \qquad m \geqq 1, \qquad (1)\\
(\gamma + \lambda)p(0, 1) &= \omega p(0, 0) + \mu p(1, 1).
\end{aligned}
$$

To solve this system of equations we introduce the probability generating function

$$G(z) = \sum p(m)z^m = G_1(z) + G_0(z),$$

where

$$G_1(z) = \sum p(m, 1)z^m, \qquad G_0(z) = \sum p(m, 0)z^m,$$

and

$$p(m) = p(m, 1) + p(m, 0).$$

Note that $G_1(1) = \sum p(m, 1)$ is simply the probability of the switch

being on and is given by $\omega/(\gamma + \omega)$. Similarly, $G_0(1) = \sum p(m, 0) = \gamma/(\gamma + \omega)$ is the probability of the switch being off. We will now derive the differential equations for G_1 and G_0 and obtain their solutions.

From (1) and the definition of $G(z)$ we have

$$\mu(z - 1)G_0'(z) + \omega G_0(z) - \gamma G_1(z) = 0,$$

$$\mu(z - 1)G_1'(z) + (\gamma + \lambda - \lambda z)G_1(z) - \omega G_0(z) = 0.$$

This system of equations is coupled. At the price of increasing the order, we can decouple the system by means of differentiation and simple substitution:

$$\mu(z - 1)G_0''(z) + [\mu + \gamma + \omega - \lambda(z - 1)]G_0'(z) - \frac{\lambda}{\mu}\omega G_0(z) = 0, \tag{2}$$

$$\mu(z - 1)G_1''(z) + [\mu + \gamma + \omega - \lambda(z - 1)]G_1'(z) - \frac{\lambda}{\mu}(\omega + \mu)G_1(z) = 0.$$

Since we are interested in the moments of the distribution of the number of busy servers, it will be convenient to obtain solutions of (2) valid about $z = 1$. Subsequently, the series can be rearranged at the origin to yield the state probabilities.

The change of variable

$$\xi = \frac{\lambda}{\mu}(z - 1)$$

transforms (2) into

$$\begin{aligned} \xi Q_0''(\xi) + (\epsilon - \xi)Q_0'(\xi) - \beta Q_0(\xi) &= 0, \\ \xi Q_1''(\xi) + (\epsilon - \xi)Q_1'(\xi) - (1 + \beta)Q_1(\xi) &= 0, \end{aligned} \tag{3}$$

where

$$\epsilon = 1 + \frac{\gamma + \omega}{\mu}, \qquad \beta = \frac{\omega}{\mu},$$

and

$$Q_1(\xi) = G_1(z), \qquad Q_2(\xi) = G_2(z).$$

Equations (3) have the solutions

$$Q_0(\xi) = C_1F_1(\beta; \epsilon; \xi),$$
$$Q_1(\xi) = D_1F_1(1 + \beta; \epsilon; \xi),$$

wbere

$$ {}_1F_1(a;b;c) = \sum_{n=0}^{\infty} \frac{(a)_n c^n}{(b)_n n!} $$

with

$$ (a)_n = a(a+1)(a+2)\cdots(a+n-1), \qquad (a)_0 = 1, $$

is the confluent bypergeometric function and C and D are arbitrary constants. The conditions $G_1(1) = \omega/(\gamma+\omega)$ and $G_0(1) = \gamma/(\gamma+\omega)$ determine C and D and hence

$$ G(z) = \frac{\gamma}{\gamma+\omega}\, {}_1F_1\left[\beta;\epsilon;\frac{\lambda}{\mu}(z-1)\right] + \frac{\omega}{\gamma+\omega}\, {}_1F_1\left[1+\beta;\epsilon;\frac{\lambda}{\mu}(z-1)\right]. \quad (4) $$

The power series in $(z-1)$ on the rigbt-band side of (4) is convergent since ${}_1F_1$ is an entire function. The factorial moments of the distribution of tbe number of busy servers, being tbe coefficients of tbe expansion (4), are simply

$$ G^{(n)}(1) = \lambda^n \frac{(\omega)_n}{(\gamma+\omega)_n}, \qquad n = 0, 1, 2, \cdots, \quad (5) $$

where we have made the normalization $\mu = 1$. For computational purposes, the following recurrence relation is useful:

$$ G^{(n+1)}(1) = \lambda \frac{(\omega+n)}{(\gamma+\omega+n)} G^{(n)}(1). \quad (6) $$

To obtain the state probabilities $p(m) = p(m,0) + p(m,1)$ we rearrange the series (4) at the origin and identify tbe coefficients in this new expansion as the state probabilities

$$ \begin{aligned} p(m) &= \frac{1}{m!} \sum_{k=0}^{\infty} G^{(m+k)}(1) \frac{(-1)^k}{k} \\ &= \frac{\lambda^m}{m!} \sum_{j=m}^{\infty} \frac{(-\lambda)^{j-m}}{(j-m)!} \frac{(\omega)_j}{(\gamma+\omega)_j}. \end{aligned} \quad (7) $$

We now show that the interrupted Poisson process provides an accurate metbod of generating overflow traffic. Let a erlangs of Poisson traffic be offered to an Erlang B system of c trunks. Let the overflow traffic be routed to an infinite trunk group and let Y be tbe number of busy servers in the infinite trunk group under statistical equilibrium. The factorial moments $M_{(n)}$ of Y and the state probabilities

$f(m) = p[Y = m]$ have been computed by Kosten:[4]

$$M_{(n)} = a^n \frac{\sigma_0(c)}{\sigma_n(c)},$$
$$f(m) = \frac{a^{c+m}}{c!m!} \sum_{k=0}^{\infty} \frac{(-a)^k}{k!\sigma_{k+m}(c)}, \tag{8}$$

where

$$\sigma_o(c) = \frac{a^c}{c!}, \qquad \sigma_j(c) = \sum_{i=0}^{c} \binom{j+i-1}{i} \frac{a^{c-i}}{(c-i)}, \qquad j = 1, 2, \cdots .$$

For a more accessible reference which gives the derivation of the factorial moments, see the appendix prepared by J. Riordan in Ref. 5.

Now consider an interrupted Poisson traffic of original intensity λ offered to an infinite trunk group, and let X be the number of busy servers under statistical equilibrium. The factorial moments of X and the state probabilities are given by (5) and (7) respectively. In this system we have three parameters, λ, γ, and ω, which are to be chosen so that the interrupted Poisson process gives the best approximation to the overflow process. In the present analysis we choose the moment approximation and, in particular, take the factorial moments. Thus we require that

$$G^{(n)}(1) = M_{(n)}, \qquad n = 1, 2, 3, \tag{9}$$

and define the error

$$E_n(a, c) = f(n) - p(n), \qquad n = 0, 1, 2, \cdots . \tag{10}$$

With the aid of the recurrence relation (6), we can express (9) as

$$\lambda\left(\frac{\omega + n}{\omega + \gamma + n}\right) = a\delta_n, \qquad n = 0, 1, 2, \tag{11}$$

where

$$\delta_n = \frac{M_{(n+1)}}{aM_{(n)}} = \frac{\sigma_n(c)}{\sigma_{n+1}(c)}.$$

Equations (11) have the solution

$$\lambda = a \frac{\delta_2(\delta_1 - \delta_0) - \delta_0(\delta_2 - \delta_1)}{(\delta_1 - \delta_0) - (\delta_2 - \delta_1)},$$
$$\omega = \frac{\delta_0}{\lambda}\left(\frac{\lambda - a\delta_1}{\delta_1 - \delta_0}\right), \tag{12}$$
$$\gamma = \frac{\omega}{a}\left(\frac{\lambda - a\delta_0}{\delta_0}\right).$$

The parameters λ, ω, and γ can be computed from (12) whenever a and c are specified. Often, however, one is concerned with final-route traffic in which only the mean, α, and variance, v, (or peakedness ratio $z = v/\alpha$) are known. There are two ways to proceed in this case.

First, using Wilkinson's equivalent random method,[5] determine S, the number of trunks, and A, the equivalent random load corresponding to the overflow traffic of mean α and variance v. Now set $a = A$ and $c = S$ and use eqs. (12) to compute λ, ω, and γ for a three-moment matcb.

A second way to proceed is to determine the equivalent random load A as before, and set $\lambda = A$ in the last two equations of (12) to compute ω and γ for a two-moment match. A satisfactory value of the equivalent random load A is given by Rapp's approximation:[6]

$$A = \alpha z + 3z(z - 1), \tag{13}$$

where z is tbe peakedness ratio v/α. In terms of α, z, and A, these equations can be written as

$$\begin{aligned} \omega &= \frac{\alpha}{A}\left(\frac{A - \alpha}{z - 1} - 1\right), \\ \gamma &= \left(\frac{A}{\alpha} - 1\right)\omega. \end{aligned} \tag{14}$$

Note that for a two-moment match it is necessary to fix one of the parameters λ, ω, or γ. However, for a positive solution, they cannot be cbosen arbitrarily–for positivity we must choose $\lambda > \alpha$, such as in (13).

To illustrate the procedure, let us take an example with overflow traffic of mean 0.61 and variance 0.95. By Wilkinson's equivalent random method, we obtain $S = 4$ trunks and $A = 3$ erlangs. The first three moments from (8) with $c = S = 4$ and $a = A = 3$ are

$$\begin{aligned} M_{(1)} &= 0.618 \\ M_{(2)} &= 0.708 \\ M_{(3)} &= 1.025. \end{aligned} \tag{15}$$

The three-moment match yields an interrupted Poisson process with parameters

$$\begin{aligned} \lambda &= 2.553 \\ \omega &= 0.646 \\ \gamma &= 2.022, \end{aligned}$$

and of course the same first three moments as in (15).

For the two-moment match, we set $\lambda = A = 3$ and from (14) obtain

$$\omega = 0.669$$
$$\gamma = 2.621 .$$

The first two moments will be equal to $M_{(1)}$ and $M_{(2)}$ of (15) and the third moment is found to be

$$G^{(3)}(1) = 1.049$$

which is not significantly different from $M_{(3)}$ in (15). Computing the state probabilities, we obtain the following result in which the two-moment match of the negative binomial fit[5] was included for comparison:

State	Exact State Probabilities	Three-Moment Match	Two-Moment Match	Negative Binomial
0	0.6164	0.6161	0.6149	0.6086
1	0.2312	0.2318	0.2344	0.2464
2	0.0965	0.0962	0.0953	0.0924
3	0.0372	0.0372	0.0366	0.0337
4	0.0130	0.0130	0.0129	0.0122
5	0.0041	0.0041	0.0042	0.0043
6	0.0012	0.0012	0.0012	0.0015
7	0.0003	0.0003	0.0003	0.0005
8	0.0001	0.0001	0.0001	0.0002

We now examine the error $E_n(a, c)$. Since little additional computation is required to obtain the three-moment match, we feel it should be done to obtain a better fit. Consequently, we examined the error for a three-moment match only. Calculations of $f(n)$, the state probabilities as they are given exactly, and $p(n)$, the interrupted Poisson approximation, have been made. This was done for groups of 1, 2, 4, 8, 16, 32, 40, and 48 trunks in the primary group with offered occupancies of 0.75 and 1, and for a group of 64 with offered occupancy of 0.75. For larger trunk groups, significant loss of accuracy prevented successful computation. We note that for the case of one trunk the approximation is exact. Where comparison could be made with previously reported results for the negative binomial and the confluent hypergeometric approximations,[7] it was found that the three-moment match using the interrupted Poisson process gave uniformly better fit to the state probabilities.

A typical result of the computations made is displayed in Table I. It was found that for a fixed occupancy the errors would increase, reach a maximum, and then decrease as the number of trunks was increased. This behavior can be seen in Fig. 1 where

$$E = \max_n |E_n(a, c)| = \max_n |f(n) - p(n)|$$

TABLE I—THREE-MOMENT MATCH
($c = 16$, $a/c = 0.75$, $\lambda = 6.83$, $\omega = 0.366$, $\gamma = 3.09$)

State n	$f(n)$	$p(n)$	$E_n(12,16) = f(n) - p(n)$	$\frac{E_n(12,16)}{f(n)}$
0	0.64866359	0.64765765	0.00100593	0.00155078
1	0.17173954	0.17395867	−0.00221912	−0.01292144
2	0.08460030	0.08381668	0.00078362	0.00926265
3	0.04523738	0.04463748	0.00059990	0.01326127
4	0.02430285	0.02417444	0.00012841	0.00528357
5	0.01280780	0.01290307	−0.00009527	−0.00743819
6	0.00655960	0.00668314	−0.00012354	−0.01883330
7	0.00325188	0.00333032	−0.00007845	−0.02412386
8	0.00155792	0.00158885	−0.00003093	−0.01985106
9	0.00072097	0.00072378	−0.00000280	−0.00388859
10	0.00032234	0.00031441	0.00000793	0.02460252
11	0.00013930	0.00013020	0.00000910	0.06530564
12	0.00005822	0.00005142	0.00000681	0.11694969
13	0.00002356	0.00001937	0.00000418	0.17756000
14	0.00000923	0.00000697	0.00000226	0.24472624
15	0.00000351	0.00000240	0.00000111	0.31590026
16	0.00000129	0.00000079	0.00000050	0.38876238
17	0.00000046	0.00000025	0.00000021	0.46060149
18	0.00000016	0.00000008	0.00000009	0.53013187
19	0.00000005	0.00000002	0.00000003	0.59587179

is plotted against c for different values of the offered occupancy a/c and again in Fig. 2 where the relative error corresponding to the state n at which $|E_n(a, c)|$ attained its maximum is plotted.

III. INTERARRIVAL TIME DISTRIBUTION

In analytical studies of systems, a description of overflow traffic is sometimes needed in terms of the interarrival time distribution.[2] This distribution is complicated and may be difficult to compute whenever the size of the trunk group which the Poisson traffic is overflowing is large.[8] If the interrupted Poisson traffic is used to generate the overflow traffic, the resulting interarrival time distribution, say $A(t)$, is simple. Indeed, we show that this distribution is given simply by the mixture of two exponential distributions:

$$A(t) = k_1(1 - e^{-r_1 t}) + k_2(1 - e^{-r_2 t}), \tag{16}$$

where

$$r_1 = \tfrac{1}{2}\{\lambda + \omega + \gamma + \sqrt{(\lambda + \omega + \gamma)^2 - 4\lambda\omega}\},$$

$$r_2 = \tfrac{1}{2}\{\lambda + \omega + \gamma - \sqrt{(\lambda + \omega + \gamma)^2 - 4\lambda\omega}\},$$

$$k_1 = \frac{\lambda - r_2}{r_1 - r_2},$$

$$k_2 = 1 - k_1.$$

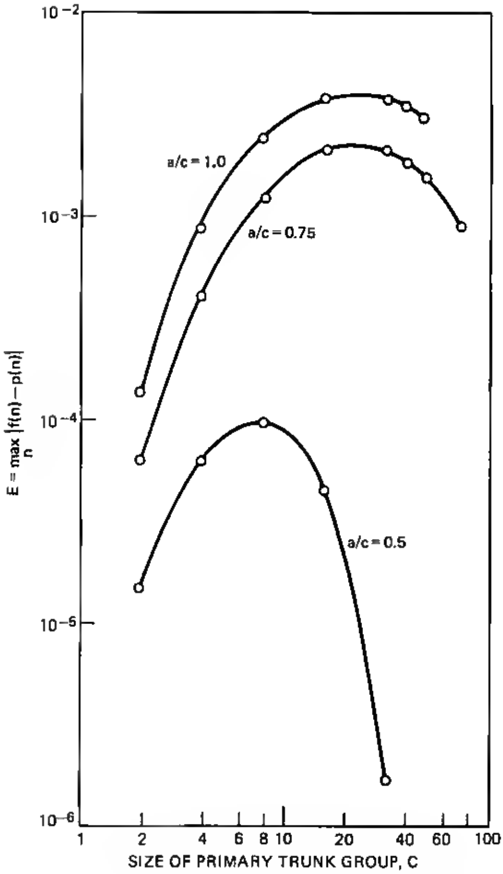

Fig. 1—Maximum absolute error E.

The proof goes as follows. Let W_n be the waiting time from $t = 0$ until the time of the nth arrival and $H_n(t)$ be the distribution of W_n. To obtain the interarrival time distribution, it is not necessary to find $H_n(t)$ for all n. The distribution $H_1(t)$ and proper choice of initial conditions at $t = 0$ is sufficient to find $A(t)$. We include the more general case here for completeness.

If $N(t)$ counts the number of arrivals in $(0, t)$ and

$$p_k(t) = P[N(t) = k],$$

then

$$H_n(t) = 1 - \sum_{k=0}^{n-1} p_k(t). \tag{17}$$

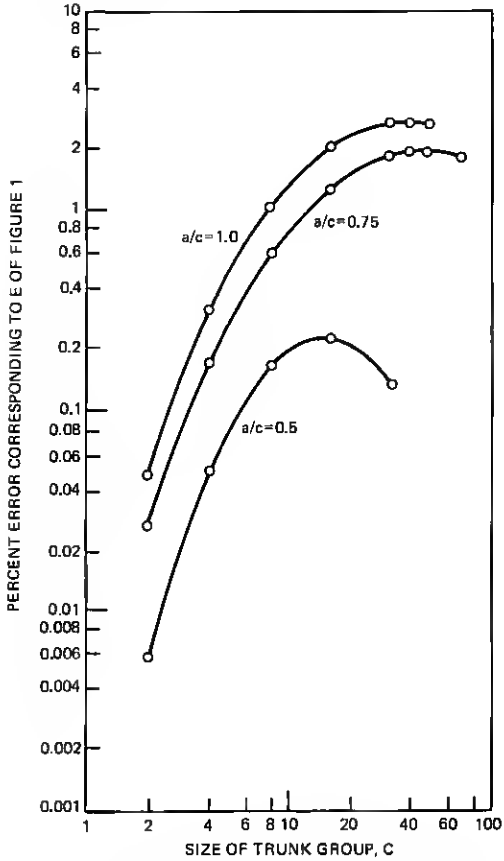

Fig. 2—Relative absolute error corresponding to Fig. 1.

This equation follows from the observation that $W_n > t$ if and only if $N(t) < n$.

Taking the Laplace-Stieltjes transform of both sides of (17) we obtain

$$\alpha_n(s) = 1 - s \sum_{k=0}^{n-1} \pi_k(s), \tag{18}$$

where

$$\alpha_n(s) = \int_0^\infty e^{-st} dH_n(t),$$

$$\pi_k(s) = \int_0^\infty e^{-st} p_k(t) dt.$$

The initial conditions used to obtain (18) are

$$p_k(0^+) = \begin{cases} 1, & k = 0 \\ 0, & k > 0. \end{cases} \tag{19}$$

We will compute $\pi_k(s)$ and hence determine $\alpha_n(s)$. From tbe expression for $\alpha_n(s)$ we then determine the interarrival distribution $A(t)$.

Let $p_{km}(t)$ be the probability tbat there were k arrivals in $(0, t)$, given that an arrival occurred at $t = 0$ and that at the instant t the switch is on if $m = 1$ and off if $m = 0$. These functions satisfy the system of differential equations

$$\begin{aligned} p'_{01}(t) &= \omega p_{00}(t) - (\lambda + \gamma)p_{01}(t), \\ p'_{k1}(t) &= \omega p_{k0}(t) - (\lambda + \gamma)p_{k1}(t) + \lambda p_{k-1,1}(t), \qquad k = 1, 2, \cdots, \\ p'_{k0}(t) &= -\omega p_{k0}(t) + \gamma p_{k1}(t), \qquad k = 0, 1, 2, \cdots, \end{aligned} \tag{20}$$

and the initial condition $p_{01}(0) = 1$.

Taking the Laplace transform of (20), we obtain

$$\begin{aligned} s\pi_{01}(s) &= \omega\pi_{00}(s) - (\lambda + \gamma)\pi_{01}(s) + 1, \\ s\pi_{k1}(s) &= \omega\pi_{k0}(s) - (\lambda + \gamma)\pi_{k1}(s) + \lambda\pi_{k-1,1}(s), \qquad k = 1, 2, \cdots, \\ s\pi_{k0}(s) &= -\omega\pi_{k0}(s) + \lambda\pi_{k1}(s), \qquad k = 0, 1, 2, \cdots, \end{aligned}$$

where

$$\pi_{kj}(s) = \int_0^\infty e^{-st} p_{kj}(t)dt.$$

This system of difference equations can be solved for $\pi_{k0}(s)$ and $\pi_{k1}(s)$ and hence $\pi_k(s)$, since $\pi_k(s)$ is the sum $\pi_{k0}(s) + \pi_{k1}(s)$. Omitting the details, we have

$$\pi_k(s) = \frac{s + \omega + \gamma}{\tau(s)} \left[\frac{\lambda(s + \omega)}{\tau(s)}\right]^k, \qquad k = 0, 1, 2, \cdots, \tag{21}$$

where

$$\tau(s) = s^2 + (\lambda + \gamma + \omega)s + \lambda\omega.$$

Substituting (21) into (18), we obtain

$$\alpha_n(s) = \left[\frac{\lambda(s + \omega)}{\tau(s)}\right]^n, \qquad n = 1, 2, \cdots. \tag{22}$$

The interarrival time distribution is given by the distribution of the

waiting time until the first arrival and hence its Laplace-Stieltjes transform is given by $\alpha_1(s)$. Inverting $\alpha_1(s)$, we obtain eq. (16).

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